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THE DIFFRACTION OF SOUND WAVES AT THE ANGULAR JOINT BETWEEN THIN ELASTIC PLATES†

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An exact solution of the boundary-contact problem of the diffraction of a plane wave at the junction of two elastic plates set at an arbitrary angle is constructed by Malyuzhinets' method. © 1997 Elsevier Science Ltd. All rights reserved.

Problems of the diffraction of sound by elastic semi-infinite plates and their joints, situated in an acoustic medium, have been considered for a long time in mechanics and acoustics, beginning with the work of Malyuzhinets [1–3] and Lamb [4], in which an exact solution of the problem of the diffraction of a plane sound wave by a thin elastic half-plane were obtained for the first time. The problem was investigated in detail in [5–8] in the case of rectangular geometry (two plates unfolded into a plane or joined at a right angle, a T-shaped junction, a rectangular volume, and so on), when the Wiener–Hopf method can be used. If the angle between the plates is not a multiple of a right angle it is not possible to obtain a solution by the Wiener–Hopf method and it is necessary to invoke a different mathematical technique, namely, the Malyuzhinets method, developed to solve problems of the diffraction and radiation of sound in angular regions of arbitrary aperture angle [9, 10]. For an angular region, the boundaries of which are absolutely rigid and slippery and covered with thin elastic plates performing only longitudinal vibrations, an exact solution of the problem of plane sound wave diffraction was obtained by Tuzhilin [11].

Below, using the general theory developed for angular regions of arbitrary aperture and for boundary conditions containing derivatives of arbitrary order [12–14], we present an exact solution of the problem of the diffraction of a plane harmonic sound wave at the angular joint of thin elastic plates which are in unilateral contact with acoustic medium and are described using fifth-order differential operators (Kirchhoff plates). Section 1 is devoted to a correct formulation of the problem, which should provide a unique solution satisfying the reciprocity principle. In Section 2, using the Malyuzhinets–Tuzhilin technique, the diffraction problem is reduced to a system of functional difference equations in Sommerfeld integral transformants, representing the sound pressure inside the acoustic medium filling the angular region. In Section 3 a general solution of the problem is obtained which contains eight undetermined constants and which satisfies the Helmholtz equation, the boundary conditions on the plates and also the conditions on the edge (the latter is understood to mean the conditions for there to be no fictitious sound sources on the edges). These constants are found from the conditions for no "parasitic" waves to exist, which would disturb the conditions at infinity (Section 4), and from the contact conditions which specify the kinematic and dynamic modes in the region where the plates are joined (Section 5). The uniqueness of the solution of the problem formulated in Section 1 and its corresponding reciprocity principle are discussed in the Appendix.

1. FORMULATION OF THE PROBLEM

Consider two thin semi-infinite plates fastened along their edges so that in a cylindrical system of coordinates (r, φ, z) in which the z axis lies along the joint line, the external sides of the plates coincide with the surfaces $\varphi = \pm\Phi$ (Fig. 1). The plates are assumed to be homogeneous, have densities $\rho_\pm$ and thicknesses $h_\pm$, and, generally speaking, do not coincide with one another (here and henceforth quantities with plus and minus subscripts relate to the upper or lower plate, respectively). The sector $|\varphi| < \Phi$ is filled with an acoustic medium of density ρ and velocity of sound c; outside the plates there is a vacuum. A plane harmonic sound wave

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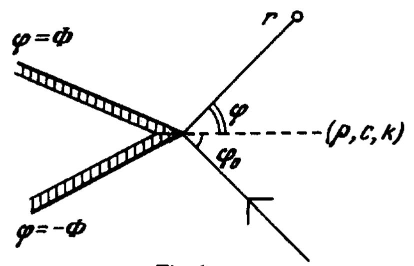

Fig. 1.

$$p_0(r,\varphi)=P_0\exp(-ikr\cos(\varphi-\varphi_0)),\quad k=\frac{\omega}{c} \tag{1.1}$$

is incident from infinity on the angular joint of the plates normal to the common edge and at an angle φ_0 to the plane $\varphi = 0$, where p_0 is the amplitude and ω is the angular frequency; the time dependence is taken in the form $\exp(-i\omega t)$. The problem consists of determining the field of the total sound pressure $p(r, \varphi)$ everywhere inside the acoustic medium.

From a mathematical point of view the problem reduces to constructing a solution of the two-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)p(r,\varphi)=0 \tag{1.2}$$

inside the plane angular sector $|\varphi| < \Phi$, $0 \leqslant r < +\infty$, with the boundary conditions

$$\mathbf{L}_\pm\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\varphi}\right)p(r,\varphi)=0,\quad \varphi=\pm\Phi \tag{1.3}$$

in which

$$\mathbf{L}_\pm\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\varphi}\right)=\mp\left(\frac{\partial^4}{\partial r^4}-\varkappa_\pm^4\right)\frac{1}{r}\frac{\partial}{\partial\varphi}+\nu_\pm,\quad \varkappa_\pm=\left(\rho_\pm h_\pm\frac{\omega^2}{D_\pm}\right)^{1/4},\quad \nu_\pm=\frac{\rho\omega^2}{D_\pm} \tag{1.4}$$

where $\varkappa_\pm$ are the wave numbers of flexural oscillations of the plates in a vacuum, and $D_\pm$ are the stiffnesses of the plates for bending. At the points where the plates are joined $r = 0$, the function $p(r, \varphi)$ must remain bounded and continuous with respect to the angle φ, and its first derivatives have only integrable singularities

$$\lim_{r\to 0}|p(r,\varphi)|=\text{const}<+\infty,\quad \lim_{r\to 0}|r\,\text{grad}\,p(r,\varphi)|=0 \tag{1.5}$$

Conditions (1.5), which will be called the conditions on the edge, arise from the requirement that there should be no fictitious sound sources on lines of geometrical and material singularities of the boundaries [15].

The conditions at infinity in problems of the diffraction of plane waves in angular regions can be formulated as the requirement for the difference between the total field $p(r, \varphi)$ and the geometrical-acoustics field $p_g(r, \varphi)$ to decrease if there is some absorption, at least as small as desired, in the medium [10], i.e.

$$\lim_{r\to+\infty}|p(r,\varphi)-p_g(r,\varphi)|=0,\quad \forall\varphi,\quad \text{Im}\,k>0 \tag{1.6}$$

The field $p_g(r, \varphi)$ in general may consist of an incident wave and its reflections from the boundaries of the region; it is constructed using the laws of geometrical acoustics and can be assumed to be known [16].

The kinematic and dynamic modes in the region where the thin plates are joined are described using the contact conditions [6]

$$\mathbf{K}_n^+p+\mathbf{K}_n^-p=0,\quad n=1,2,3,4 \tag{1.7}$$

$$\mathbf{K}_n^{\pm} p = \lim_{r\to 0}\left(\mathbf{H}_{1n}^{\pm}\left(\frac{\partial}{\partial r}\right)\frac{1}{r}\,\frac{\partial p}{\partial \varphi}(r,\pm\Phi) + \mathbf{H}_{2n}^{\pm}\left(\frac{\partial}{\partial r}\right)p(r,\pm\Phi)\right)$$

where $\mathbf{H}_{mn}^{\pm}(\partial/\partial r)$ are polynomials and $m = 1, 2$. Conditions (1.7) contain the normal components of the displacement vectors

$$\xi_{\pm}(r) = \mp\frac{1}{\omega^2 \rho r}\,\frac{\partial p}{\partial \varphi}(r,\pm\Phi)$$

of points of the plates, the angles of emergence of the plates from the joint $\Phi_{\pm} = \pm(\Phi - \xi'_{\pm}(0))$, the moments $M_{\pm} = \pm D_{\pm}\xi''_{\pm}(0)$ and the normal components of the forces $F_{\pm} = -D_{\pm}\xi'''_{\pm}(0)$ (the primes denote derivatives with respect to the r coordinate). It follows from the uniqueness theorem (Appendix) that the nature of the relationships between these quantities in contact conditions (1.7) must not contradict the overall requirement

$$U_+ + U_- \geqslant 0 \quad (U_{\pm} = D_{\pm}\,\mathrm{Im}(\xi'_{\pm}(0)\bar{\xi}''_{\pm}(0) - \xi_{\pm}(0)\bar{\xi}'''_{\pm}(0))) \tag{1.8}$$

which arises from the energy conservation law (the bar denotes complex conjugation). Moreover, for any pair of parameters φ_1 and φ_2 having the meaning of angles of incidence, the following relation must be satisfied

$$J_+(\varphi_1,\varphi_2) + J_-(\varphi_1,\varphi_2) = 0 \tag{1.9}$$

$$(J_{\pm}(\varphi_1,\varphi_2) = D_{\pm}(\eta_{\pm}(\varphi_1,\varphi_2) - \eta_{\pm}(\varphi_1,\varphi_2)))$$

$$(\eta_{\pm}(x,y) = \xi_{\pm}(0,x)\xi'''_{\pm}(0,y) - \xi'_{\pm}(0,x)\xi''_{\pm}(0,y))$$

which imposes one more constraint on the relation between the limit values of the parameters characterizing the plate joining conditions (here, for convenience, the corresponding value of the angle of incidence φ_0 is indicated as the second argument of the functions $\xi_{\pm}(r,\varphi_0)$). Condition (1.9) is a consequence of the reciprocity principle (see the Appendix).

It is obvious that the correct formulation of the problem of diffraction by the corner joints of thin elastic plates must imply that the contact conditions (1.7) are formulated appropriately and should not contradict the general relations (1.8) and (1.9). It can be shown by direct substitution that the type of contact conditions most often encountered, namely

$$\xi_{\pm}(0) = 0, \quad \xi'_{\pm}(0) = 0 \ \text{(rigid clamping)} \tag{1.10}$$

$$\xi_{\pm}(0) = 0, \quad M_{\pm} = 0 \ \text{(hinged joint)} \tag{1.11}$$

$$M_{\pm} = 0, \quad F_{\pm} = 0 \ \text{(free edge)} \tag{1.12}$$

$$\xi_{\pm}(0) = 0, \quad \Phi_+ - \Phi_- = 2\Phi, \quad D_+\xi''_+(0) = D_-\xi''_-(0) \ \text{(rigid joint)} \tag{1.13}$$

convert relations (1.8) and (1.9) into identities. Hence, the use of any of these guarantees that the corresponding boundary-contact problem has only one solution, and this solution does not contradict the reciprocity principle.

2. CONVERSION TO FUNCTIONAL EQUATIONS

We will construct the solution of the problem in the form of a Sommerfeld integral [9]

$$p(r,\varphi) = \frac{P_0}{2\pi i}\int_{\gamma} \exp(-ikr\cos\alpha)S(\alpha+\varphi)d\alpha \tag{2.1}$$

which, as can be shown by direct substitution, is a particular solution of the Helmholtz equation (1.2) for an arbitrary choice of the function $S(\alpha)$—the transformant of the Sommerfeld integral. The contour of integration γ consists of two symmetrical loops $\gamma_\pm$ in the complex plane α (Fig. 2), the ends of which are arranged such that integral (2.1) converges. It is assumed that there is not a single singularity of the function $S(\alpha + \varphi)$ inside the loop $\gamma_\pm$ when $|\varphi| \leqslant \Phi$. It must be emphasized that the possibility of representing the required solution by integral (2.1) can be justified for a wide range of diffraction problems, which, in the final analysis, arises from the equivalence of the Malyuzhinets integral transformation and the Laplace transformation [17].

The transformant $S(\alpha)$ will be sought in the class of meromorphic functions, having a single simple pole with unit residue in the band $\Pi_0 = \{\alpha: |\operatorname{Re}\alpha| < \Phi\}$ at the point $\alpha = \varphi_0$ and which satisfies the condition $|S(\alpha + \varphi) - S(\pm\infty)| \to 0$ uniformly with respect to φ as $\operatorname{Im}\alpha \to \pm\infty$ which ensures the separation of the specified incident wave (1.1) as $r \to +\infty$ and the satisfaction of conditions (1.5) on the edge, respectively. Note that, using the transformation $S(\alpha) \to S(\alpha) + \text{const}$, under which the Sommerfeld integrals are invariant, by an appropriate choice of the constant one can always endeavour to satisfy the relation

$$S(+i\infty) = -S(-i\infty) \tag{2.2}$$

which will also be assumed below.

Substituting integral (2.1) into the boundary conditions (1.3) we obtain a system of two integral identities

$$\int_\gamma \exp(-ikr\cos a)L_\pm(\alpha)S(\alpha \pm \Phi)d\alpha = 0, \quad 0 \leqslant r < +\infty \tag{2.3}$$

in which the functions $L_\pm(\alpha) = \mathbf{L}_\pm(-ik\cos\alpha, -ik\sin\alpha)$ can naturally be called the symbols of the boundary operator; using (1.4) they can be written in the form

$$L_\pm(\alpha) = \pm ik\sin\alpha(k^4\cos^4\alpha - \varkappa_\pm^4) + \nu_\pm \tag{2.4}$$

As we know [9, 17], the necessary and sufficient condition for integrals (2.3) to vanish is that the odd part of the integrand should be equal to a certain trigonometric polynomial

$$L_\pm(\alpha)S(\alpha \pm \Phi) - L_\pm(-\alpha)S(-\alpha \pm \Phi) = 2\sin\alpha \sum_{n=1}^{N_\pm} C_n^\pm \cos^{n-1}\alpha \tag{2.5}$$

with arbitrary coefficients $C_n^\pm$, which are independent of α (by making the replacement of variable $q = -ik\cos\alpha$ this assertion can be reduced to the generalized Liouville theorem). The orders of the polynomials $N_\pm$ are determined by the behaviour of the required solution and the symbols of the boundary operators as $\operatorname{Im}\alpha \to \infty$; hence, in the case considered we must have $N_\pm \leqslant 5$. From the

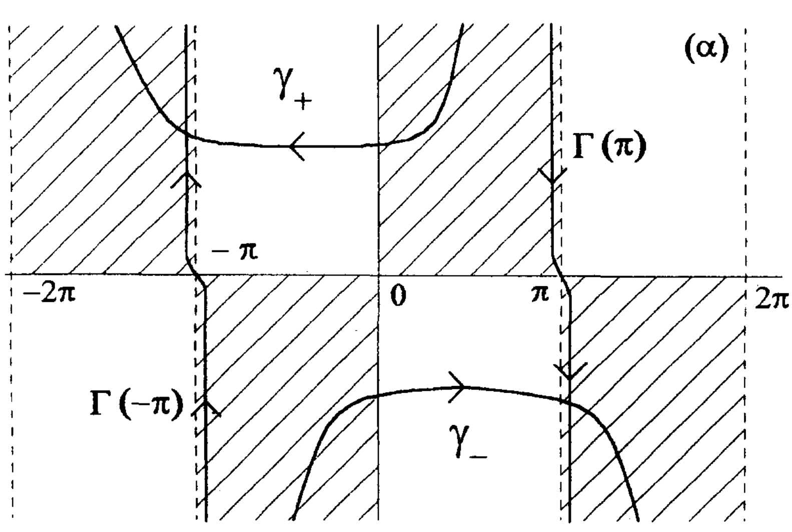

Fig. 2.

equations of system (2.5) as $\text{Im}\,\alpha \to \infty$ it follows that, without loss of generality, the constants $C_5^{\pm}$ can be equated to zero, since they are proportional to the sum of the limit values of $S(+i\infty) + S(-i\infty)$, equal to zero by virtue of the normalization condition (2.2). Thus, the satisfaction of boundary conditions (1.3) by integral (2.1) has been reduced to a system of two functional difference equations (2.5), in which the right-hand side may contain eight arbitrary constants $C_n^{\pm}$ ($n = 1, 2, 3, 4$).

3. SOLUTION OF THE FUNCTIONAL EQUATIONS

To solve the functional equations (2.5) we will use a method which was proposed and used for the first time in [10] when considering the diffraction by a wedge with impedance boundaries. The method consists of converting the initial system with variable coefficients to a system with constant coefficients by making the replacement

$$S(\alpha) = \Psi(\alpha)S_1(\alpha) \tag{3.1}$$

where $S_1(\alpha)$ is a new unknown function, while the function $\Psi(\alpha)$ is an auxiliary function and is chosen as a certain particular solution of system (2.5) with zero right-hand side. Then, for $S_1(\alpha)$ the problem reduces to a system of inhomogeneous functional equations with constant coefficients

$$S_1(\alpha \pm \Phi) - S_1(-\alpha \pm \Phi) = f_{\pm}(\alpha) \tag{3.2}$$

$$f_{\pm}(\alpha) = \frac{2\sin\alpha}{L_{\pm}(\alpha)\Psi(\alpha \pm \Phi)} \sum_{n=1}^{4} C_n^{\pm} \cos^{n-1}\alpha$$

The equations for the function $\Psi(\alpha)$ can also be reduced to a system with constant coefficients if we introduce its logarithmic derivative $Y(\alpha) = \Psi'(\alpha)/\Psi(\alpha)$

$$Y(\alpha \pm \Phi) + Y(-\alpha \pm \Phi) = \pm\frac{R'_{\pm}(\alpha)}{R_{\pm}(\alpha)}, \quad R_{\pm}(\alpha) = -\frac{L_{\pm}(\mp\alpha)}{L_{\pm}(\pm\alpha)} \tag{3.3}$$

It can be verified that the functions $R_{\pm}(\alpha)$ are identical with the reflection coefficients when a plane wave, incident at a glancing angle α, is reflected from an infinite plate [18]. Hence, the numbers $\alpha = \mp\alpha_n^{\pm}$, where $\alpha_n^{\pm}$ are the zeros of the symbols of the boundary operators, have the meaning of the Brewster angles (generally speaking complex) for plates which form boundary surfaces: the reflection coefficient of a plane wave incident on them at these angles is zero. It can be seen from (2.4) that the functions $L_{\pm}(\alpha)$ may be written as trigonometric polynomials of the fifth degree in $\sin\alpha$, and hence in the complex plane α all the roots of these functions can be expressed in terms of five roots, which lie in any strip of width π. We will choose the strip $|\,\text{Re}\,\alpha\,| \leq \pi/2$ as this strip, while the roots which belong to it will be denoted by $\alpha = \mp\alpha_n^{\pm}$ ($n = 1, 2, \dots, 5$). It is obvious that the numbers $\theta_n^{\pm}$ can be found using the relations $\theta_n^{\pm} = \arcsin\beta_n^{\pm}$, $\text{Re}\,\theta_n^{\pm} \in (-\pi/2, \pi/2)$, where $\beta_n^{\pm}$ are the roots of the characteristic equations

$$\beta^5 - 2\beta^3 + a_{\pm}\beta + ib_{\pm} = 0 \quad \left(a_{\pm} = 1 - \left(\frac{\varkappa_{\pm}}{k}\right)^4, \quad b_{\pm} = \frac{\nu_{\pm}}{k^5}\right) \tag{3.4}$$

The solution of system (3.3), which has no zeros and no poles in the strip Π_0, is constructed in explicit form using a Fourier transformation of the function $Y(\alpha)$, which leads to the algebraic problem of determining its transformants. As a result, the following representation can be obtained for the auxiliary function

$$\Psi(\alpha) = \Psi_+(\alpha)\Psi_-(\alpha) \tag{3.5}$$

$$\Psi_{\pm}(\alpha) = \prod_{n=1}^{5} \left\{\psi_{\Phi}\left(\alpha \pm \Phi + \frac{\pi}{2} - s_n^{\pm}\theta_n^{\pm}\right)\psi_{\Phi}\left(\alpha \pm -\frac{\pi}{2} + s_n^{\pm}\theta_n^{\pm}\right)\right\}^{s_n^{\pm}}$$

$$s_n^{\pm} = \text{sign}(\text{Re}\,\theta_n^{\pm}), \quad n = 1, 2, \dots, 5$$

where $\psi_{\Phi}(\alpha)$ is a special Malyuzhinets function [10], for which, when deriving (3.5), the following integral representation was used in [19]

$$\psi_\Phi(\alpha)=\exp\left\{-\frac{1}{2}\int_0^{+\infty}\frac{\mathrm{ch}(\alpha t)-1}{t\,\mathrm{ch}(\pi t/2)\mathrm{sh}(2t\Phi)}dt\right\}$$

The function $\psi_\Phi(\alpha)$ has no zeros or poles in the strip $|\operatorname{Re}\alpha|<\pi/2+2\Phi$, and hence expression (3.5) is in fact a function which possesses a similar property in the strip Π_0. The fact that the function $\Psi(\alpha)$ satisfies the homogeneous system (3.5) can also be shown by direct substitution if we write the following expansion for the system coefficients

$$L_\pm(\alpha)=\pm ik^5\prod_{n=1}^{5}(\sin\alpha\pm\sin\theta_n^\pm)$$

and we take into account the properties of the Malyuzhinets function [10].

The form of (3.5) depends on the arrangement of the numbers $\theta_n^\pm$ in the complex plane, more accurately on the signs of their real parts. Hence, we need to consider the arrangement of the roots of Eq. (3.4) in the complex plane β. Analysis shows that for real coefficients its roots lie symmetrically about the imaginary axis, and it has one and only one purely imaginary root with a negative imaginary part. For example, for a thin steel plate in water, the roots of the equation are situated approximately at the points of intersection of a circle of radius $|\beta|=|b_\pm|^{1/5}$ with rays emerging from the point $\beta=0$ and making angles with the negative part of the imaginary axis that are multiples of $2\pi/5$ (Fig. 3). For other combinations of the plate and acoustic medium parameters (for example, steel and air) the approximate expression for the roots and their arrangement in the complex plane may be different, but for the theory developed here it is only important that two roots of the characteristic equation (3.4) (in the notation employed these are the roots $\beta_{2,4}^\pm$) should have a negative real part, and the roots $\beta_{3,5}^\pm$ should have a positive real part. For the real coefficients of the equation the sign of the real part of the numbers is indefinite since they are pure imaginary. Henceforth, when constructing the solution of the system of functional equations (2.5), it will be more convenient to assume that the numbers $\beta_1^\pm$ nevertheless have a certain small, but non-zero, positive real part. This will be the case, for example, if we make the coefficients $b_\pm$ complex: $b_\pm=|b_\pm|\exp(i\arg b_\pm)$, where $0<\arg b_\pm\ll 1$. In the final formulae we can revert to pure real coefficients by letting $\arg b_\pm$ tend to zero.

With these assumptions, for the parameters $s_n^\pm$ in (3.5) we obtain $s_n^\pm=(-1)^{n-1}$ ($n=1,2,\dots,5$), which completely defines the form and properties of the function $\Psi(\alpha)$. In particular, as $\operatorname{Im}\alpha\to\infty$, in view of the asymptotic properties of the Malyuzhinets function for large and complex arguments [10, 20]

$$\psi_\Phi(\alpha)=\frac{1}{\sqrt{2}}\psi_\Phi\left(\frac{\pi}{2}\right)\exp\left(-i\frac{\pi\alpha}{8\Phi}\operatorname{sign}(\operatorname{Im}\alpha)\right)(1+o(1))$$

we have

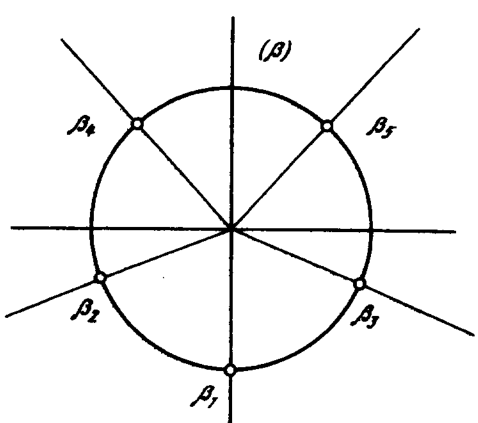

Fig. 3.

$$\Psi(\alpha)=\frac{1}{4}\psi_\Phi^4\left(\frac{\pi}{2}\right)\exp(-i\mu\alpha\,\mathrm{sign}(\mathrm{Im}\,\alpha))(1+o(1)),\quad \mu=\frac{\pi}{2\Phi} \tag{3.6}$$

We will now construct the function $S_1(\alpha)$. The general solution of system (3.2) must be constructed from the particular solution of the inhomogeneous system and, possibly, the general solution of the corresponding homogeneous equations [21]. The particular solution of the inhomogeneous problem, analytic in the strip Π_0, is found using a Fourier transformation as

$$\Lambda(\alpha)=\Lambda_+(\alpha)+\Lambda_-(\alpha) \tag{3.7}$$

$$\Lambda_\pm(\alpha)=\pm\frac{i}{8\Phi}\int_{-i\infty}^{+i\infty}\mathrm{tg}\left(\frac{\pi}{4\Phi}(\alpha+\beta\pm\Phi)\right)f_\pm(\beta)d\beta$$

The integrals $\Lambda_\pm(\alpha)$ converge absolutely, since, in view of estimate (3.6), the functions $f_\pm(\beta)$ decrease more rapidly than $\exp(-|\mu|\ \mathrm{Im}\ \beta\ |)$ as $\mathrm{Im}\ \beta\to\pm\infty$. As $\mathrm{Im}\ \alpha\to\pm\infty$ the function is described by the asymptotic formula

$$\Lambda(\alpha)=\pm\frac{i}{4\Phi}\exp(\pm i\mu\alpha)\int_{-i\infty}^{+i\infty}\exp(i\mu\beta)(f_+(\beta)+f_-(\beta))d\beta(1+o(1)) \tag{3.8}$$

which can be obtained if in the integrands of the integrals $\Lambda_\pm(\alpha)$ we replace the tangent function by the two leading terms of its expansion as $\mathrm{Im}\ \alpha\to\pm\infty$ and take into account the fact that the functions $f_\pm(\beta)$ are odd. The function $\Lambda(\alpha)$ can be continued into the exterior of the strip Π_0 either directly, using the integral formula (3.7), or using the functional equations

$$\Lambda(\alpha\pm\Phi)-\Lambda(-\alpha\pm\Phi)=f_\pm(\alpha) \tag{3.9}$$

which follow from (3.2).

The function $\Lambda(\alpha)$ has no poles in the strip Π_0, which follows from the convergence of the integrals $\Lambda_\pm(\alpha)$ in this strip. Hence, in order to obtain a pole in the solution at the point $\alpha=\varphi_0$ we introduce into consideration the meromorphic function

$$\sigma(\alpha,\varphi_0)=\frac{\mu\cos(\mu\varphi_0)}{\sin(\mu\alpha)-\sin(\mu\varphi_0)} \tag{3.10}$$

which has unit residue at the point $\alpha=\varphi_0$ and at the same time is a solution of the homogeneous system (3.2) (this can be proved directly). The function

$$S_2(\alpha)=\frac{\sigma(\alpha,\varphi_0)}{\Psi(\varphi_0)}+\Lambda(\alpha) \tag{3.11}$$

will then be a solution of system (3.2) with the required singularity when $\alpha=\varphi_0$.

It follows from (3.8) and (3.10) that as $\mathrm{Im}\ \alpha\to\infty$ the function $S_2(\alpha)$ decreases as $O(\exp(-|\mu|\ \mathrm{Im}\ \alpha\ |))$, which, taking into account the asymptotic form (3.6), ensures that the product $\Psi(\alpha)S_2(\alpha)$ is bounded in the neighbourhood of an infinitely distant point in the complex plane α. On the other hand, by construction, the function $\Psi(\alpha)S_2(\alpha)$ has only a single pole inside the strip Π_0 situated at the point $\alpha=\varphi_0$, and has a residue equal to unity. Hence, by substituting the function $S_2(\alpha)$ into (3.1) as the function $S_1(\alpha)$ we obtain the required solution of the system of functional equations (2.5), possessing the required analytic properties.

The homogeneous equations (3.2) also have other solutions which differ from the function $\sigma(\alpha,\varphi_0)$, which can be divided into two types—meromorphic and integer (the latter are simply linear combinations of cosines $\cos(\mu n(\alpha-\Phi))$ ($n=0,1,2,\ldots$)) [21]. The solution of the homogeneous system (3.2) remains its solution after adding or subtracting the solutions of the homogeneous equations, and also after multiplying or dividing by these solutions. However, its analytic properties are then changed: poles either appear or disappear and the behaviour in the neighbourhood of an infinitely distant point becomes different. It can be shown that one cannot supplement the function $S_2(\alpha)$ with any other solutions of the homogeneous equations either additively or multiplicatively without disturbing its decrease as $\mathrm{Im}\ \alpha\to\infty$ or without introducing into the strip Π_0 poles that do not correspond to the postulated conditions.

Thus, the expression

$$S(\alpha) = \Psi(\alpha)S_2(\alpha) \tag{3.12}$$

gives the required solution of the functional equations in the most general form (the function $\Psi(\alpha)$ is defined by (3.5) while $S_2(\alpha)$ is defined by (3.11)). It contains eight undetermined constants $C^{\pm}{}_n$ (n = 1, 2, 3, 4). For arbitrary values of these constants, integral (2.1) with transformant (3.12) is a solution of the Helmholtz equation (1.2) everywhere inside the angular region $|\phi| < \Phi$, satisfies the boundary conditions (1.3) and (1.4) on its sides, separates the incident wave (1.1) (in the "illuminated" part of the space $|\varphi - \varphi_0| < \pi$), and is also a bounded and continuous function at the point $r = 0$. For the value of the sound pressure on the rib, as a simple consequence of the asymptotic forms (3.6) and (3.8) and the Malyuzhinets formula [22]

$$\lim_{r\to 0}(r,\varphi) = iP_0(S(i\infty) - S(-i\infty)) \tag{3.13}$$

we can write the accurate representation

$$\lim_{r\to 0}(r,\varphi) = P_0\psi_\Phi^4\left(\frac{\pi}{2}\right)\left(\frac{\mu\cos(\mu\varphi_0)}{\Psi(\varphi_0)} - \frac{1}{8\Phi}\int_{-i\infty}^{+i\infty}\exp(i\mu\beta)(f_+(\beta) + f_-(\beta))d\beta\right)$$

which is independent of the direction in which the observation point tends to the corner point.

The constants $C_n^{\pm}$ in the general solution are found from the remaining conditions: the conditions at infinity (1.6) and the contact conditions (1.7).

4. THE POLES OF THE TRANSFORMANT AND THE CONDITIONS AT INFINITY

We will consider the behaviour of integral (2.1) as $r \to +\infty$. By continuous deformation the contours $\gamma_\pm$ can be converted into two parallel contours $\Gamma(\pm\pi)$, passing along the lines Re $\alpha = \pm\pi$ in the complex plane α (Fig. 2). By choosing these contours such that they are shifted slightly (as small as desired) into regions where the condition Im cos $\alpha < 0$ is satisfied (shown hatched), we obtain integrals which tend to zero as $r \to +\infty$ since their integrands tend to zero at all points of the integration contours $\Gamma(\pm\pi)$. During the deformation the poles of the function $S(\alpha + \varphi)$ may intersect, if they fall inside the regions bounded above and below by the contours $\gamma_\pm$, and on the right and left by the contours $\Gamma(\pm\pi)$. If among these there are poles which are situated in the unhatched parts of the region indicated, then in the estimate of integral (2.1) terms that increase exponentially as $r \to +\infty$ appear as residues in these poles, which leads to a violation of the condition at infinity (1.6). Hence, only those poles of the transformant $S(\alpha)$ which are situated in the strip $\Pi = \{\alpha: |\operatorname{Re}\alpha| \leq \pi + \Phi\}$ can affect the satisfaction of the condition at infinity. Those which lead to the occurrence of terms which increase as $r \to +\infty$ must be eliminated from the solution found (these poles will henceforth be called "forbidden" poles).

We will now consider the poles of expression (3.12). The poles of the function $\sigma(\alpha, \varphi_0)$ (see formula (3.10)) form two families

$$\alpha_j = \begin{cases}\varphi_0 + 4j\Phi \\ 2\Phi - \varphi_0 + 4j\Phi\end{cases}, \quad j = 0, \pm 1, \pm 2, \ldots \tag{4.1}$$

The position of the poles of these families in the complex plane depends on the value of the angle of incidence φ_0, and among these there is a pole $\alpha = \varphi_0$, corresponding to the incident wave (1.1). The residue in this and the remaining poles of the families (4.1) contain no constants $C^{\pm}{}_n$ and make the contributions

$$p_j^g(r,\varphi) = P_0\frac{\Psi(\alpha_j)\cos(\mu\varphi_0)}{\Psi(\varphi_0)\cos(\mu\alpha_j)}\exp(-ikr\cos(\alpha_j - \varphi)) \tag{4.2}$$

These contributions are identical both in phase and amplitude with plane waves reflected from the sides of the region (the amplitude factor in (4.2) can be converted to reflection coefficients or their products in the case of multiple rereflections, which may occur in narrow angular regions $\Phi < \pi/2$, using the well-known functional properties of Malyuzhinets functions in exactly the same way as was done for impedance boundary conditions in [16]). Hence, all the poles of the function $\sigma(\alpha, \varphi_0)$ turn out to be permissible; being incident in the strip Π they lead to separation of the residues which describe the waves of geometrical acoustics when estimating the integral (2.1) far from the line along which the plates are joined.

We will write the functions $\Psi_\pm(\alpha)$ in (3.5) as

$$\Psi_\pm(\alpha)=\Psi_1^\pm(\alpha)\frac{\Psi_3^\pm(\alpha)\Psi_5^\pm(\alpha)}{\Psi_2^\pm(\alpha)\Psi_4^\pm(\alpha)} \tag{4.3}$$

$$\Psi_q^\pm(\alpha)=\psi_\Phi\left(\alpha\pm\Phi+\frac{\pi}{2}+(-1)^q\theta_q^\pm\right)\psi_\Phi\left(\alpha\pm\Phi-\frac{\pi}{2}-(-1)^q\theta_q^\pm\right)$$

It follows from (3.5) and (4.3) that the poles of the function $\Psi(\alpha)$ coincide with the poles of the functions $\Psi^\pm{}_q(\alpha)$ when $q = 1, 3, 5$, and with their zeros when $q = 2, 4$. As we know [10], the Malyuzhinets function $\psi_\Phi(\alpha)$ when $\alpha = \pm\alpha_{nm}$, where $\alpha_{nm} = \pi(2m-1)/2 + 2\Phi(2n-1)$ $(n, m = 1, 2, 3, \ldots)$, has zeros if m is odd and has poles if m is even. Hence, the poles of the function $\Psi(\alpha)$ can be found from the relations

$$\alpha=\begin{cases}-\Phi-\pi/2+\chi\theta_q^+\pm\alpha_{nm}\\ -\Phi+\pi/2-\chi\theta_q^+\pm\alpha_{nm}\\ \Phi-\pi/2+\chi\theta_q^-\pm\alpha_{nm}\\ \Phi+\pi/2-\chi\theta_q^-\pm\alpha_{nm}\end{cases},\quad \chi=\begin{cases}1 & \text{for } q=1,3,5,\quad m=2,4,6,\ldots\\ -1 & \text{for } q=2,4,\quad m=1,3,5,\ldots\end{cases}\quad n=1,2,3,\ldots \tag{4.4}$$

Before analysing the arrangement of the poles of the function $\Psi(\alpha)$ in the complex plane, we will show that the function $\Lambda(\alpha)$, which is given by (3.7) and occurs in the expression for the transformant (3.12) in terms of the function $S_2(\alpha)$, introduces no new poles into the solution, and all the singularities of the function $S(\alpha)$ are exhausted by the poles of the functions $\sigma(\alpha, \varphi_0)$ and $\Psi(\alpha)$.

In order to show this we will compare the analytical properties of the functions $\Psi(\alpha)$ and $\Omega(\alpha) = \Psi(\alpha)\Lambda(\alpha)$. By construction (see Section 3) these functions are analytic in the strip Π_0 and, correspondingly, having no poles in this strip, are solutions of the functional equations

$$L_\pm(\alpha)\Psi(\alpha\pm\Phi)-L_\pm(-\alpha)\Psi(-\alpha\pm\Phi)=0 \tag{4.5}$$

$$L_\pm(\alpha)\Omega(\alpha\pm\Phi)-L_\pm(-\alpha)\Omega(-\alpha\pm\Phi)=2\sin\alpha\sum_{n=1}^{4}C_n^\pm\cos^{n-1}\alpha$$

(the coefficients $L_\pm(\alpha)$ are described by (2.4)). The functions $\Psi(\alpha)$, $\Omega(\alpha)$ are continued into the exterior of the strip Π_0 by means of relations (4.5). For example, in the strip $\Pi_1 = \{\alpha\colon \Phi < \operatorname{Re}\alpha \leqslant 3\Phi\}$ these functions can be found from the formulae

$$\Psi(\alpha+2\Phi)=\frac{L_+(-\alpha-\Phi)}{L_+(\alpha+\Phi)}\Psi(-\alpha),\quad \alpha\in\Pi_0$$

$$\Omega(\alpha+2\Phi)=\frac{1}{L_+(\alpha+\Phi)}\left(L_+(-\alpha-\Phi)\Omega(-\alpha)+2\sin(\alpha+\Phi)\sum_{n=1}^{4}C_n^+\cos^{n-1}(\alpha+\Phi)\right)$$

Both in this and the other case the poles of both functions in the strip Π_1 coincide with the zero of the same function $L_+(\alpha+\Phi)$. We can similarly compare the singularities of the functions $\Psi(\alpha)$, $\Omega(\alpha)$ in any other strip in the complex plane α and show that their poles coincide (although the residues in these poles may be different). Hence it follows that, despite the fact that the function $\Lambda(\alpha)$ is meromorphic (this can be shown, for example, using its integral representation (3.7), by continuing it into the exterior of the strip Π_0), each pole necessarily coincides with some of the zeros of the function $\Psi(\alpha)$ and, as a result, the function $\Lambda(\alpha)$ introduces no new poles different from (4.4) into the product $\Psi(\alpha)\Lambda(\alpha)$ and correspondingly into the function $S(\alpha)$.

We will determine which of the poles of the function $\Psi(\alpha)$ can fall within the strip Π. We first note that the poles corresponding $m \geqslant 3$ in (4.4) lie outside this strip, an d hence the analysis can be confined to the cases $m = 1, 2$. It can further be shown that when $q = 1, 3, 5$ not one of the poles of family (4.4) falls within Π, which is due to the presence of a positive real part in the numbers $\theta_q^{\pm}$. As a result, it turns out that only poles of family (4.4) corresponding to $q = 2, 4$ and $m = 1$ can be situated inside the strip considered. The number of poles in Π depends on the aperture of the angular region Φ, which determines the shift between poles with different values of n: the smaller the value of the parameter Φ the closer the poles to one another and the greater the number of them that can lie inside Π.

Bearing in mind the signs of the imaginary parts $\operatorname{Im}\theta_2^{\pm} < 0$, $\operatorname{Im}\theta_4^{\pm} > 0$ (see Section 3), it can be shown that of all the poles situated in the strip Π, the forbidden poles, i.e. those the residues of which lead to an increase in the solution as $r \to +\infty$ and to a violation of the condition at infinity (1.6), will be the poles which belong to one of the families listed below

$$\alpha_{pn}^f = \begin{cases} \theta_2^+ + \pi + \Phi + 4n\Phi, & p=1 \\ -\theta_2^+ - \pi - 3\Phi - 4n\Phi, & p=2 \\ -\theta_4^+ + \Phi + 4n\Phi, & p=3 \\ \theta_4^+ - 3\Phi - 4n\Phi, & p=4 \end{cases}, \quad \alpha_{pn}^f = \begin{cases} -\theta_2^- - \pi - \Phi - 4n\Phi, & p=5 \\ \theta_2^- + \pi + 3\Phi + 4n\Phi, & p=6 \\ \theta_4^- - \Phi - 4n\Phi, & p=7 \\ -\theta_4^- + 3\Phi + 4n\Phi, & p=8 \end{cases} \tag{4.6}$$

$$n = 0, 1, 2, \ldots, N_p$$

(the number of poles in a family with number p is equal to $N_p + 1$ and depends on the particular values of the parameters Φ and $\operatorname{Re}\theta_{2,4}^{\pm}$). The residues of integral (2.1) in the remaining poles of the transformant $S(\alpha)$ from the strip Π tend to zero as $r \to +\infty$ and do not violate the conditions at infinity.

In order to formulate the conditions which ensure the required behaviour of the solution as $r \to +\infty$, we will first consider a simpler case when the aperture angle between the plates satisfies the inequality $\Phi > \pi/2$. There can then be eight poles inside the strip Π, namely

$$\alpha = \Phi - \theta_q^+, \quad \Phi + \pi + \theta_q^+, \quad -\Phi - \pi - \theta_q^-, \quad -\Phi + \theta_q^-, \quad q = 2, 4$$

of which the following four will be forbidden

$$\alpha = \alpha_{10}^f, \quad \alpha_{30}^f, \quad \alpha_{50}^f, \quad \alpha_{70}^f \tag{4.7}$$

In order to ensure the correct behaviour of the solution as $r \to +\infty$ we need to equate the residues of the function $S(\alpha)$ in the poles (4.7) to zero. Since these poles are poles of the function $\Psi(\alpha)$, the corresponding conditions can be written as the conditions for the function $S_2(\alpha)$, which is the factor after $\Psi(\alpha)$ in the general solution (3.12), to vanish, which, using (3.11), gives

$$\sigma(\alpha_{p0}^f, \varphi_0)/\Psi(\varphi_0) + \Lambda(\alpha_{p0}^f) = 0, \quad p = 1, 3, 5, 7 \tag{4.8}$$

It can be seen that conditions (4.8) have the form of linear algebraic equations in the unknown constants $C_n^{\pm}$ $(n = 1, 2, 3, 4)$. In the complete system of eight equations, which is constructed in this investigation and which can be conveniently written in matrix form

$$\mathbf{E}\mathbf{C} = \mathbf{g} \tag{4.9}$$

$$\mathbf{C} = (C_1^+, C_2^+, C_3^+, C_4^+, C_1^-, C_2^-, C_3^-, C_4^-)^T, \quad \mathbf{g} = (g_1, g_2, \ldots, g_8)^T$$

$$\mathbf{E} = \begin{Vmatrix} E_{11}^+ & E_{12}^+ & E_{13}^+ & E_{14}^+ & E_{11}^- & E_{12}^- & E_{13}^- & E_{14}^- \\ E_{21}^+ & E_{22}^+ & E_{23}^+ & E_{24}^+ & E_{21}^- & E_{22}^- & E_{23}^- & E_{24}^- \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{81}^+ & E_{82}^+ & E_{83}^+ & E_{84}^+ & E_{81}^- & E_{82}^- & E_{83}^- & E_{84}^- \end{Vmatrix}$$

the first four rows of system (4.9) correspond to conditions (4.8). For the first four elements of the vector g, by (4.8) we obtain

$$g_p = -\sigma(\alpha_p^f, \varphi_0)/\Psi(\varphi_0), \quad p = 1, 2, 3, 4$$

$$\alpha_1^f = \alpha_{10}^f, \quad \alpha_2^f = \alpha_{30}^f, \quad \alpha_3^f = \alpha_{50}^f, \quad \alpha_4^f = \alpha_{70}^f$$

The elements of the first four rows of matrix E are derived from representation (3.7); here one must take into account that the poles α_{p0}^f lie outside the strip Π_0, and for the function $\Lambda(\alpha)$ it is necessary to use expressions obtained by analytical continuation into the corresponding parts of the complex plane α. In addition, when writing the final formulae we can remove the auxiliary assumption that a small positive imaginary part is present in the coefficients $b_\pm$ in the characteristic equations (3.4), which was made from considerations of convenience when constructing the general solution of the problem (see Section 3). When taking the limit as $\arg b_\pm \to 0$ the poles of the integrands in integrals (3.7), related to the zeros of the functions $L_\pm(\beta)$ at the points $\beta = \mp\theta_1^\pm$, reach the integration contour, which leads to the occurrence of additional semiresidue terms. The following formulae can then be obtained for the elements of the first four rows of the matrix **E**

$$E_{1n}^+ = Q_n^+(\pi+\theta_2^+) + X_n^+ \operatorname{ctg}\left(\frac{\mu}{2}(\pi+\theta_1^+ +\theta_2^+)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{ctg}\left(\frac{\mu}{2}(\pi+\theta_2^+ -\beta)\right) Q_n^+(\beta)\,d\beta$$

$$E_{1n}^- = X_n^- \operatorname{tg}\left(\frac{\mu}{2}(\pi+\theta_2^+ -\theta_1^-)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{tg}\left(\frac{\mu}{2}(\pi+\theta_2^+ -\beta)\right) Q_n^-(\beta)\,d\beta$$

$$E_{2n}^+ = -Q_n^+(\theta_4^+) + X_n^+ \operatorname{ctg}\left(\frac{\mu}{2}(\theta_1^+ -\theta_4^+)\right) - \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{ctg}\left(\frac{\mu}{2}(\theta_4^+ +\beta)\right) Q_n^+(\beta)\,d\beta \tag{4.10}$$

$$E_{2n}^- = -X_n^- \operatorname{tg}\left(\frac{\mu}{2}(\theta_4^+ +\theta_1^-)\right) - \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{tg}\left(\frac{\mu}{2}(\theta_4^+ +\beta)\right) Q_n^-(\beta)\,d\beta$$

$$E_{3n}^+ = X_n^+ \operatorname{tg}\left(\frac{\mu}{2}(\pi-\theta_1^+ +\theta_2^-)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{tg}\left(\frac{\mu}{2}(\pi+\theta_2^- +\beta)\right) Q_n^+(\beta)\,d\beta$$

$$E_{3n}^- = -Q_n^-(\pi+\theta_2^-) + X_n^- \operatorname{ctg}\left(\frac{\mu}{2}(\pi+\theta_1^- +\theta_2^-)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{ctg}\left(\frac{\mu}{2}(\pi+\theta_2^- +\beta)\right) Q_n^-(\beta)\,d\beta$$

$$E_{4n}^+ = -X_n^+ \operatorname{tg}\left(\frac{\mu}{2}(\theta_1^+ +\theta_4^-)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{tg}\left(\frac{\mu}{2}(-\theta_4^- +\beta)\right) Q_n^+(\beta)\,d\beta$$

$$E_{4n}^- = -Q_n^-(-\theta_4^-) + X_n^- \operatorname{ctg}\left(\frac{\mu}{2}(\theta_1^- -\theta_4^-)\right) + \frac{i}{8\Phi}\int_{-i\infty}^{+i\infty} \operatorname{ctg}\left(\frac{\mu}{2}(-\theta_4^- +\beta)\right) Q_n^-(\beta)\,d\beta$$

$$Q_n^\pm(\beta) = \frac{2\sin\beta\cos^{n-1}\beta}{L_\pm(\beta)\Psi(\beta\pm\Phi)}, \quad X_n^\pm = \frac{\mu}{2}\,\frac{\sin\theta_1^\pm\cos^{n-1}\theta_1^\pm}{L_\pm'(\mp\theta_1^\pm)\Psi(\mp\theta_1^\pm\pm\Phi)}, \quad n=1,\ 2,\ 3,\ 4$$

All the integrals in (4.10) are taken in the sense of the principal value at the points $\beta = \mp\theta_1^\pm$ which lie on the integration contour. These integrals converge absolutely as $\beta \to \pm i\infty$, since the function $\Psi(\alpha)$ has the estimate (3.6) as $\operatorname{Im}\alpha \to \pm\infty$. The matrix **E** acquires certain symmetry properties if the plates which form the angle are the same. Then $L_+(\alpha) = L_-(-\alpha)$, $\theta_n^+ = \theta_n^-$, $\Psi(\alpha)$, $Q_n^+(-\beta) = -Q_n^-(\beta)$, $X_n^+ = -X_n^-$, which leads to the relations

$$E_{1n}^\pm + E_{3n}^\mp = 0, \quad E_{2n}^\pm + E_{4n}^\mp = 0, \quad n=1,\ 2,\ 3,\ 4$$

We will now consider the case when the aperture angle of the region satisfies the condition $\Phi < \pi/2$. Then additional poles, which differ from those described by (4.7), appear in integral (2.1) inside the unhatched half-strip (Fig. 2); they all belong to the families (4.6). It is important, however, that this should not lead to additional conditions imposed on the residues of the transformant at these poles. It can be shown that if the constants $C_n^\pm$ are chosen from conditions (4.8), this will automatically ensure that the residues of the function $S(\alpha)$ will also vanish at those additional poles which occur inside the unhatched parts of the complex plane α on changing from the case $\Phi > \pi/2$ to the case $\Phi < \pi/2$.

For example, suppose $\pi/4 < \Phi < \pi/2$. Then, in addition to the poles (4.7) two more poles become forbidden: at the point $\alpha = \alpha^f_{40}$ and $\alpha = \alpha^f_{80}$. However, it turns out that at these points the functions from (3.11) take the same values as at the points $\alpha = \alpha^f_{30}$ and $\alpha = \alpha^f_{70}$, namely

$$\sigma(\alpha^f_{40},\ \varphi_0) = \sigma(\alpha^f_{30},\ \varphi_0),\quad \sigma(\alpha^f_{80},\ \varphi_0) = \sigma(\alpha^f_{70},\ \varphi_0)$$

$$\Lambda(\alpha^f_{40}) = \Lambda(\alpha^f_{30}),\quad \Lambda(\alpha^f_{80}) = \Lambda(\alpha^f_{70})$$

which ensures that the residues $S(\alpha)$ at the poles α^f_{40} and α^f_{80} are zero if the residues at the poles α^f_{30} and α^f_{70}, respectively, are equal to zero. Note that the equalities for the function $\sigma(\alpha, \varphi_0)$ arise from the elementary properties of trigonometric functions, while the equalities for the function $\Lambda(\alpha)$ arise from the well-known properties of the Malyuzhinets function and the functional equations (3.9), which it satisfies.

We will calculate, for example, the value of the function $\Lambda(\alpha)$ at the point $\alpha = \alpha^f_{80} = 3\Phi - \theta^-_4$. By making the replacement $\alpha \to \alpha + 2\Phi$ in the first of Eqs (3.9), we obtain the expression

$$\Lambda(\alpha + 3\Phi) = f_+(\alpha + 2\Phi) + \Lambda(-\alpha - \Phi)$$

which, using the second equation of the same system, can be written as

$$\Lambda(\alpha + 3\Phi) = f_+(\alpha + 2\Phi) - f_-(\alpha) + \Lambda(\alpha - \Phi)$$

The substitution $\alpha = -\theta^-_4$, taking into account the fact that the function $f_-(\alpha)$ is odd, and also the identity

$$f_+(2\Phi - \theta^-_4) = 0$$

which follows from the fact that one of the Malyuzhinets functions in the function $\Psi^-_4(3\Phi - \theta^-_4)$ vanishes (see representation (4.3)), leads to the formula

$$\Lambda(\alpha^f_{80}) = f_-(\theta^-_4) + \Lambda(-\theta^-_4 - \Phi)$$

where the right-hand side is none other than $\Lambda(\alpha^f_{70})$, by virtue of the second equation of system (3.9). In a similar way we can derive the second of the above relations for the function $\Lambda(\alpha)$.

We can similarly verify that the solution of the problem in which the constants $C^\pm_n$ are chosen using the four conditions (4.8), remains correct not only when $\pi/4 < \Phi < \pi/2$ but also when $\pi/6 < \Phi < \pi/4$. There is obviously no need to check this property again later since it reflects the analytical nature of the dependence of the solution of the diffraction boundary-value problem on the parameter Φ: the solution obtained for a certain non-zero range of values of this parameter (here $\pi/2 < \Phi < \pi$) can be continued analytically into the whole remaining range of values of this parameter. From the physical point of view, the fact that the residues are equal to zero at all the remaining poles (4.6) merely denotes that these residues describe all possible rereflections with respect to the sides of the corner region of the space considered of a certain four fundamental wave processes, corresponding to the residues of the poles (4.7). Since these, the last ones, are not excited, by virtue of conditions (4.8), their rereflections are correspondingly also not excited (as regards rereflections inside corner regions with impedance boundary conditions see [16]).

Thus, the conditions at infinity (1.6) can be satisfied using the four conditions (4.8), which are imposed on the eight constants $C^\pm_n$ ($n = 1, 2, 3, 4$), contained in the general solution of the problem. The remaining four conditions required for a unique determination of all the constants must be obtained from the contact conditions (1.7).

5. SATISFACTION OF THE CONTACT CONDITIONS

Using the general solution given in Section 3, we will calculate the values of the physical parameters $\xi_\pm(0), \Phi_\pm, M_\pm, F_\pm$ occurring in the contact conditions (1.7), which will subsequently expressed in terms of the limiting values of the quantities

$$\frac{d^m}{dr^m}\left(\frac{1}{r}\ \frac{\partial p}{\partial \varphi}(r,\ \pm\Phi)\right),\quad m = 0,\ 1,\ 2,\ 3,\quad \text{as}\ r \to 0$$

Differentiating integral (2.1) we obtain, after changing to integration along the contour γ_+

$$\frac{d^m}{dr^m}\left(\frac{1}{r}\ \frac{\partial p}{\partial \varphi}(r,\ \pm\Phi)\right) = \frac{(-ik)^{m+1}}{2\pi i} P_0 \times \tag{5.1}$$

$$\times \int_{\gamma_+} \exp(-ikr\cos\alpha)\sin\alpha\cos^m\alpha(S(\alpha\pm\Phi)+S(-\alpha\pm\Phi))d\alpha, \quad m=0,\ 1,\ 2,\ 3$$

The limiting values of these integrals as $r \to 0$ are governed by the behaviour of the integrand at the ends of the integration contour as $\operatorname{Im}\alpha \to +\infty$ (see formula (3.13)).

In order to estimate this behaviour, we will use the functional equations (2.5), which the function $S(\alpha)$ satisfies, and we will write

$$S(\alpha\pm\Phi)+S(-\alpha\pm\Phi)=S(\alpha\pm\Phi)\left(1+\frac{L_\pm(\alpha)}{L_\pm(-\alpha)}\right)-\frac{2\sin\alpha}{L_\pm(-\alpha)}\sum_{n=1}^{4}C_n^\pm\cos^{n-1}\alpha \tag{5.2}$$

As $\operatorname{Im}\alpha \to +\infty$, the symbols of the boundary operators have the estimates

$$\frac{1}{L_\pm(\alpha)}=\mp\frac{i}{k^5\sin\alpha\cos^4\alpha}(1+O(\exp(-4\operatorname{Im}\alpha))) \tag{5.3}$$

Hence $1 + L_\pm(\alpha)/L_\pm(-\alpha) = O(\exp(-5\operatorname{Im}\alpha))$, and the first term on the right-hand side of (5.2) makes no contribution to the limiting values of integrals (5.1), since, in view of the fact that the function $S(\alpha)$ is bounded as $\operatorname{Im}\alpha = \infty$ and is analytic inside the loop γ_+, when integrating this term the contour γ_+ can be contracted to a point. Simplifying the integrand in the remaining term using (5.3), we can obtain

$$\lim_{r\to 0}\frac{d^m}{dr^m}\left(\frac{1}{r}\frac{\partial p}{\partial\varphi}(r,\ \pm\Phi)\right)=\mp\frac{(-ik)^{m+1}}{\pi k^5}P_0\sum_{n=1}^{4}C_n^\pm\lim_{r\to 0}I_{mn}(r), \quad m=0,\ 1,\ 2,\ 3 \tag{5.4}$$

$$I_{mn}(r)=\int_{\gamma_+}\exp(-ikr\cos\alpha)\sin\alpha\cos^{n+m-5}\alpha d\alpha=\begin{cases}-\dfrac{2\pi i(-ikr)^{4-n-m}}{(4-n-m)!}, & 1\leqslant m+n\leqslant 4\\ 0, & m+n\geqslant 5\end{cases}$$

(the integrals $I_{mn}(r)$ are evaluated by making the replacement of variable $t = \cos\alpha$ and using the theorem of residues). Taking the limit in (5.4) we obtain

$$\lim_{r\to 0}\frac{d^m}{dr^m}\left(\frac{1}{r}\frac{\partial p}{\partial\varphi}(r,\ \pm\Phi)\right)=\pm 2(-i)^m k^{m-4}C_{4-m}^\pm P_0, \quad m=0,\ 1,\ 2,\ 3$$

Hence, each of the constants $C_n^\pm$ has a clear physical meaning

$$C_4^\pm=-\frac{\omega^2\rho k^4}{2P_0}\xi_\pm(0),\quad C_3^\pm=\frac{\omega^2\rho k^3}{2iP_0}\xi_\pm'(0)$$

$$C_2^\pm=\pm\frac{\omega^2\rho k^2}{2D_\pm P_0}M_\pm,\quad C_1^\pm=\frac{\omega^2\rho k}{2iD_\pm P_0}F_\pm,$$

defining the values of the displacements, the angles of emergence, the moments of the forces and the forces on the edges of the plates.

These relations show that the contact conditions (1.7) reduce to linear algebraic equations in the constants $C_n^\pm$ ($n = 1, 2, 3, 4$) and in general can be written as the four lower rows of the matrix equation (4.9). For example, for conditions (1.10)–(1.13) we can obtain $C_3^\pm = 0$, $C_4^\pm = 0$ (rigid clamping), $C_2^\pm = 0$, $C_4^\pm = 0$ (a hinged joint), $C_1^\pm = 0$, $C_2^\pm = 0$ (free ends), and $D_+C_2^+ = D_-C_2^-$, $C_3^+ = -C_3^-$, $C_4^\pm = 0$ (a rigid joint). Hence, for these methods of fastening the edges of the plates to one another, the contact conditions can be reduced to very simple equations for the constants $C_n^\pm$. Using these equations we can eliminate the four unknown constants, and system (4.9) can be reduced to a system of four linear algebraic equations in the remaining four constants C_n. The matrix elements in this system can be written in terms of integrals (4.10), which can be obtained numerically and in some cases analytically.

After determining all the constants $C_n^\pm$, the integral formula (2.1) with the transformant (3.12) gives the required solution of the problem, which will be unique for the specified incident wave (1.1), the chosen model of the plates (1.4) and the method by which they are fastened, fixed in the contact conditions (1.7). For the contact conditions, without contradicting relation (1.9), this solution must satisfy the reciprocity relation (in the sense of symmetry with respect to the angles of incidence φ_0 and observation φ of the wave pattern scattered at the junction).

We would expect the above solution to be an effective method in investigations of the diffraction and scattering of sound waves by shells of complex shape, situated in an acoustic medium, since it requires solutions of a linear algebraic system of no higher than the eighth order, contains only the well-known special functions $\psi_\Phi(\alpha)$ (the Malyuzhinets functions, see, for example, [10, 20]), and can be written in the form of a Sommerfeld integral, for which methods of investigation have been developed both in the near field ($kr < 1$) [23], and in the far field ($kr > 1$) [10, 11, 16, 17, 24].

6. APPENDIX

The uniqueness conditions. We will show that the diffraction problem formulated in Section 1, has a unique solution when $0 < \arg k < \pi/2$ for real values of the plate parameters, if condition (1.8) is satisfied. We will assume in the derivation that all the derivatives of the required solution encountered in the boundary conditions and the Helmholtz equation have meaning.

We will assume that the problem has one more solution $p_1(r, \varphi)$, which satisfies the same conditions as the function $p(r, \varphi)$. The function $u(r, \varphi) = p(r, \varphi) - p_1(r, \varphi)$ must then satisfy the Helmholtz equation (1.2), the boundary conditions (1.3) and (1.4), the conditions on the rib (1.5), and the contact conditions (1.7), and must tend to zero as $r \to +\infty$, since the fields of geometrical acoustics for the functions $p(r, \varphi)$ and $p_1(r, \varphi)$ must obviously be identical. We multiply the Helmholtz equation for the function $u(r, \varphi)$ by the complex-conjugate function $u(r, \varphi)$ and integrate the expression obtained over the plane bounded region $S_{\varepsilon R}$, the boundary of which $C_{\varepsilon R}$ consists of sections which pass along these actual boundary surfaces $\varphi = \pm\Phi$ and the arc of circles of small radius ε and large radius R with centre at the point $r = 0$. Using Green's formula and separating the imaginary part in the result, we obtain the relation

$$\operatorname{Im} k^2 \int_{\varepsilon}^{R} \int_{-\Phi}^{\Phi} |u(r, \varphi)|^2 \, r dr d\varphi + \operatorname{Im} \oint_{C_{\varepsilon R}} \bar{u}(r, \varphi) \frac{\partial u}{\partial n}(r, \varphi) d\tau = 0 \tag{6.1}$$

where $\partial/\partial n$ denotes the operation of differentiation along the direction of the outward normal to the region $S_{\varepsilon R}$, which $d\tau$ is an element of length of the contour $C_{\varepsilon R}$. In (6.1) we can take the limits as $\varepsilon \to 0$, $R \to +\infty$, since the function $u(r, \varphi)$ satisfies the conditions on the rib and at infinity uniformly with respect to the angular coordinate φ. The limiting relation contains contour integrals only along real boundaries $\varphi = \pm\Phi$, $0 \leq r < +\infty$, where we can write $\partial/\partial n = \pm r^{-1} 2\partial/\partial\varphi$; now introducing the normal components of the displacement vectors of points on the plates

$$\xi_\pm(r) = \mp \frac{1}{\omega^2 \rho r} \frac{\partial u}{\partial \varphi}(r, \pm\Phi) \tag{6.2}$$

we obtain

$$\frac{\operatorname{Im} k^2}{\omega^2 \rho} \int_0^{+\infty} \int_{-\Phi}^{+\Phi} |u(r, \varphi)|^2 \, r dr d\varphi = \operatorname{Im} \int_0^{+\infty} (\bar{u}(r, \Phi)\xi_+(r) + \bar{u}(r, \Phi)\xi_-(r)) dr \tag{6.3}$$

Using the boundary conditions we express the functions $u(r, \pm\Phi)$ in terms of the corresponding displacements

$$u(r, \pm\Phi) = -D_\pm \left(\frac{d^4}{dr^4} - \varkappa_\pm^4 \right) \xi_\pm(r)$$

we substitute these relations into (6.3) and carry out integration by parts twice. This gives the equation

$$\frac{\operatorname{Im} k^2}{\omega^2 \rho} \int_0^{+\infty} \int_{-\Phi}^{+\Phi} |u(r, \varphi)|^2 \, r dr d\varphi + U_+ + U_- = 0 \tag{6.4}$$

in which the terms $U_\pm$ outside the integrals are defined by the expressions in parentheses in (1.8). If the kinematic and dynamic plate contact conditions at the point $r = 0$ are such that inequality (1.8) is satisfied, it follows from (6.4) that the function $u(r, \varphi)$ can only be identically zero everywhere in the region, which also proves the uniqueness of the solution of the diffraction problem in question.

The reciprocity relation. Suppose $G(\mathbf{r}, \mathbf{r}_0)$ is Green's function of the problem formulated in Section 1, i.e. the function which is a solution of the inhomogeneous Helmholtz equation with the delta-function $\delta(\mathbf{r} - \mathbf{r}_0)$ on the right-hand side, and satisfies the boundary conditions (1.3) and (1.4), the conditions on the rib (1.5), the contact conditions (1.7), and also tends to zero at infinity if $\operatorname{Im} k > 0$. The reciprocity property of the solution of the diffraction problem is usually taken to mean symmetry of Green's function with respect to a permutation of its arguments—the coordinates of the source and of the observation point

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1) \tag{6.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are two arbitrary points of the region.

We will choose two functions $v(r, \varphi) = G(\mathbf{r}, \mathbf{r}_1)$ and $w(r, \varphi) = G(\mathbf{r}, \mathbf{r}_2)$ and apply to them the second Green's formula

$$\int\limits_{S_{\varepsilon R}} (v\,\Delta w - w\Delta v)\,rdrd\varphi = \int\limits_{C_{\varepsilon R}} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n}\right)d\tau \tag{6.6}$$

in which the form of the region $S_{\varepsilon R}$ and its boundaries $C_{\varepsilon R}$ were described above when deriving the uniqueness condition. Taking the Helmholtz equation into account, which the functions $v(r, \varphi)$ and $w(r, \varphi)$ satisfy, and also the conditions on the edge and an infinity, we obtain from (6.6)

$$G(\mathbf{r}_2, \mathbf{r}_1) - G(\mathbf{r}_1, \mathbf{r}_2) = I_+ + I_- \left(I_\pm = \pm\int\limits_0^{+\infty}\left(v\frac{\partial w}{\partial \varphi} - w\frac{\partial v}{\partial \varphi}\right)\Big|_{\varphi=\pm\Phi}\frac{dr}{r}\right) \tag{6.7}$$

Relation (6.5) will obviously be satisfied if the expression on the right-hand side of (6.7) is equal to zero. Using the boundary conditions, which the functions $v(r, \varphi)$ and $w(r, \varphi)$ satisfy when $\varphi = \pm\Phi$, and integrating by parts, the integrals $I_\pm$ can be reduced to the form

$$I_\pm = \frac{1}{\mathrm{v}_\pm}\lim_{r\to 0}(V_\pm W_\pm''' - V_\pm' W_\pm'' + V_\pm'' W_\pm' - V_\pm''' W_\pm)$$

$$\left(V_\pm = \frac{1}{r}\frac{\partial v}{\partial \varphi}(r, \pm\Phi), \quad W_\pm = \frac{1}{r}\frac{\partial w}{\partial \varphi}(r, \pm\Phi)\right)$$

and we then change from Green's functions $v(r, \varphi)$, $w(r, \varphi)$ to the solutions for plane waves by taking the limit

$$p(r, \varphi, \varphi_m) = \lim_{r_m\to\infty} 4i\frac{G(r, \varphi, r_m, \varphi_m)}{H_0^{(1)}(kr_m)}, \quad m = 1, 2,$$

where $H_0^{(1)}(kr_m)$ is the Hankel function. Hence the reciprocity relation leads to a condition which the solutions of the problem of plane-wave diffraction must satisfy for arbitrary values of the angles of incidence φ_1 and φ_2. By changing to the components of the displacement vector (6.2) we obtain (1.9).

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